# Five-loop renormalization group functions of $O(n)$-symmetric $\phi^4$-theory and $\epsilon$-expansions of critical exponents up to $\epsilon^5$

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**Abstract**

Motivated by the discovery of errors in six of the 135 diagrams in the published five-loop expansions of the $\beta$-function and the anomalous dimensions of the $O(n)$-symmetric $\phi^4$-theory in $D = 4 - \epsilon$ dimensions we present the results of a full analytic reevaluation of all diagrams. The divergences are removed by minimal subtraction and $\epsilon$-expansions are given for the critical exponents $\eta$, $\nu$, and $\omega$ up to order $\epsilon^5$.

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1) During the last two decades, much effort has been invested into studying the scalar quantum field theory with $\phi^4$-interaction. On the one hand, such a theory describes correctly many experimentally observable features of critical phenomena. Field theoretic renormalization group techniques [1] in $D = 4 - \epsilon$ dimensions [2, 3, 4] combined with Borel resummation methods of the resulting $\epsilon$-expansions [5] led to extremely accurate determinations of the critical exponents of all $O(n)$ universality classes. The latter requires the knowledge of the asymptotic behaviour of perturbation series in four dimensions which is completely known in this theory [6]. Apart from such important applications, the $\phi^4$-theory, being the simplest renormalizable quantum field theory in the four dimensional space-time, has been an ideal ground for testing new methods of calculating Feynman diagrams and for studying the structure of perturbation theory.

The RG functions of the $\phi^4$-theory were first calculated analytically in four dimensions using dimensional regularization [7] and the minimal subtraction (MS) scheme [8] in the three- and four-loop approximations in Refs. [9] and [10]. The critical exponents were obtained as $\epsilon$-expansions [3] up to terms of order $\epsilon^3$ and $\epsilon^4$.

The five-loop anomalous dimension of the field $\phi$ and the associated critical exponent $\eta$ to order $\epsilon^5$ were determined analytically in [11]. The five-loop $\beta$-function and the anomalous dimension of the mass were given in Ref. [12]. However, three of the 124 four-point diagrams contributing to the $\beta$-function at the five-loop level could be evaluated only numerically. The analytic calculation of the $\beta$-function was finally completed in [13]. The ensuing $\epsilon$-expansions for the critical exponents were obtained up to order $\epsilon^5$ in [14].

Intending further applications, the Berlin group of the authors undertook an independent recalculation of the perturbation series of Refs. [11, 12], using the same techniques, and discovered errors in six of the 135 diagrams. This meant that the subsequent results of [13, 14] were also incorrect. When visiting the Moscow group the errors were confirmed and we can now jointly report all expansions in the correct form.

2) We consider the $O(n)$-symmetric theory of $n$ real scalar fields $\phi^a$ $(a = 1, 2, \ldots, n)$ with the Lagrangian

$$L = \frac{1}{2}\partial_\mu \phi^a \partial_\mu \phi^a + \frac{m_B^2}{2}\phi^a\phi^a + \frac{16\pi^2}{4!} g_B (\phi^a\phi^a)^2 \tag{1}$$

in an euclidean space with $D = 4 - \epsilon$ dimensions. The bare (unrenormalized) coupling

constant $g_{\rm B}$ and mass $m_{\rm B}$ are expressed via renormalized ones as

$$g_{\rm B} = \mu^\epsilon Z_g g = \mu^\epsilon \frac{Z_4}{(Z_2)^2} g, \qquad m_{\rm B}^2 = Z_{m^2} m^2 = \frac{Z_{\phi^2}}{Z_2} m^2 \ . \tag{2}$$

Here $\mu$ is the unit of mass in dimensional regularization and $Z_4$, $Z_2$, $Z_{m^2}$ are the renormalization constants of the vertex function, propagator and mass, respectively, with $Z_{\phi^2}$ being the renormalization constant of the two-point function obtained from the propagator by the insertion, in all possible ways, of the vertex $(-\phi^2)$ [10]. In the MS-scheme the renormalization constants do not depend on dimensional parameters and are expressible as series in $1/\epsilon$ with purely $g$-dependent coefficients:

$$Z_i = 1 + \sum_{k=1}^{\infty} \frac{Z_{i,k}(g)}{\epsilon^k} \ . \tag{3}$$

The $\beta$-function and the anomalous dimensions entering the RG equations are expressed in the standard way as follows:

$$\beta(g) = \frac{\epsilon}{2} g + \left. \frac{d\ g}{d\ \ln \mu^2} \right|_{g_{\rm B}} = \frac{1}{2} g \frac{\partial\ Z_{g,1}}{\partial\ g} \ , \tag{4}$$

$$\gamma_m = \left. \frac{d\ \ln m}{d\ \ln \mu} \right|_{g_{\rm B}} = - \frac{d\ \ln Z_{m^2}}{d\ \ln \mu^2} = \frac{1}{2} g \frac{\partial\ Z_{m^2,1}}{\partial\ g} \ , \tag{5}$$

$$\gamma_i(g) = \left. \frac{d\ \ln Z_i}{d\ \ln \mu^2} \right|_{g_{\rm B}} = - \frac{1}{2} g \frac{\partial\ Z_{i,1}}{\partial\ g} \ , \quad i = 2, 4, \phi^2 \ . \tag{6}$$

We also use the relations

$$\beta(g) = g[2\gamma_2(g) - \gamma_4(g)] \ , \quad \gamma_m(g) = \gamma_2(g) - \gamma_{\phi^2}(g), \tag{7}$$

which follow from the relations between renormalization constants implied by (2) and are useful for the calculations of $\beta(g)$ and $\gamma_m(g)$.

To determine all RG functions up to five loops we calculate the five-loop approximation to the three constants $Z_2$, $Z_4$ and $Z_{\phi^2}$. The constant $Z_2$ contains the counterterms of the 11 five-loop propagator diagrams. Two of them were calculated erroneously in Ref. [11]. The constant $Z_4$ receives contributions from 124 vertex diagrams. Of these diagrams, 90 contribute to $Z_{\phi^2}$ after appropriate changes of combinatorial factors. Four of the 124 counterterms were calculated erroneously in Ref. [12].

In the present paper we have used the same methods as in the previous works [11, 12] to calculate the counterterms from the dimensionally regularized Feynman integrals, namely, the method of infrared rearrangement [15], the Gegenbauer

polynomial $x$-space technique (GPXT) [16], the integration-by-parts algorithm [17], and the $R^*$-operation [18]. Three diagrams were calculated analytically first in [13] by using the so-called method of uniqueness, later the same results were obtained for them by using the Gegenbauer polynomials in $x$-space together with several non-trivial tricks [19]. A detailed description of the calculations including the diagramwise results will be presented in a separate publication.

The analytic results of our recalculation of the five-loop approximations to the RG functions $\beta(g)$, $\gamma_2(g)$ and $\gamma_m(g)$ are [$\zeta(n)$ is the Riemann $\zeta$-function]:

$$
\begin{aligned}
\beta(g) \;=\; & \tfrac{g^2}{6}[n+8] \;-\; \tfrac{g^3}{6}[3n+14] \\
& +\tfrac{g^4}{432}\;[33n^2+922n+2960\;+\;\zeta(3)\cdot 96(5n+22)] \\
& -\tfrac{g^5}{7776}\;[-5n^3+6320n^2+80456n+196648 \\
& \qquad +\zeta(3)\cdot 96(63n^2+764n+2332) \\
& \qquad -\zeta(4)\cdot 288(5n+22)(n+8) \\
& \qquad +\zeta(5)\cdot 1920(2n^2+55n+186)] \\
& +\tfrac{g^6}{124416}[13n^4+12578n^3+808496n^2+6646336n+13177344 \\
& \qquad +\zeta(3)\cdot 16(-9n^4+1248n^3+67640n^2+552280n+1314336) \\
& \qquad +\zeta^2(3)\cdot 768(-6n^3-59n^2+446n+3264) \\
& \qquad -\zeta(4)\cdot 288(63n^3+1388n^2+9532n+21120) \\
& \qquad +\zeta(5)\cdot 256(305n^3+7466n^2+66986n+165084) \\
& \qquad -\zeta(6)(n+8)\cdot 9600(2n^2+55n+186) \\
& \qquad +\zeta(7)\cdot 112896(14n^2+189n+526)]\;,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\gamma_2(g) \;=\; & \tfrac{g^2}{36}(n+2) \;-\; \tfrac{g^3}{432}(n+2)[n+8] \\
& +\tfrac{g^4}{5184}(n+2)\qquad [5(-n^2+18n+100)] \\
& -\tfrac{g^5}{186624}(n+2)[39n^3+296n^2+22752n+77056 \\
& \qquad -\zeta(3)\cdot 48(n^3-6n^2+64n+184) \\
& \qquad +\zeta(4)\cdot 1152(5n+22)]\;,
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\gamma_m(g) \;=\; & \tfrac{g}{6}(n+2) \;-\; \tfrac{g^2}{36}(n+2)[5] \;+\; \tfrac{g^3}{72}(n+2)[5n+37] \\
& -\tfrac{g^4}{15552}(n+2)\,[-n^2+7578n+31060 \\
& \qquad +\zeta(3)\cdot 48(3n^2+10n+68)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \zeta(4)\cdot 288(5n+22)] \\
&+\tfrac{g^5}{373248}(n+2)[21n^3+45254n^2+1077120n+3166528 \\
&\qquad\qquad + \zeta(3)\cdot 48(17n^3+940n^2+8208n+31848) \\
&\qquad\qquad - \zeta^2(3)\cdot 768(2n^2+145n+582) \\
&\qquad\qquad + \zeta(4)\cdot 288(-3n^3+29n^2+816n+2668) \\
&\qquad\qquad + \zeta(5)\cdot 768(-5n^2+14n+72) \\
&\qquad\qquad + \zeta(6)\cdot 9600(2n^2+55n+186)]\ .
\end{aligned}
\tag{10}
$$

For $n=1$ the series have the numerical form:

$$\beta(g) = 1.5\,g^2 - 2.833\,g^3 + 16.27\,g^4 - 135.8\,g^5 + 1424.2841\,g^6\ , \tag{11}$$

$$\gamma_2 = 0.0833\,g^2 - 0.0625\,g^3 + 0.3385\,g^4 - 1.9256\,g^5\ , \tag{12}$$

$$\gamma_m = 0.5\,g - 0.4167\,g^2 + 1.75\,g^3 - 9.978\,g^4 + 75.3778\,g^5\ . \tag{13}$$

Note that the five-loop coefficients have changed by about 0.3 % for the $\beta$-function, by about 9 % for $\gamma_m$, and by a factor of three for $\gamma_2$ in comparison with the wrong results of Refs. [11, 12].

3) These RG functions can now be used to calculate the critical exponents describing the behaviour of a statistical system near the critical point of the second order phase transition [4]. At the critical temperature $T = T_\mathrm{C}$, the asymptotic behaviour of the correlation function for $|\mathbf{x}| \to \infty$ has the form

$$\Gamma(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{D-2+\eta}}\ . \tag{14}$$

Close to $T_\mathrm{C}$, the correlation length behaves for $t = T - T_\mathrm{C} \to 0$ as

$$\xi \sim t^{-\nu}(1 + \mathrm{const}\cdot t^{\omega\nu} + \ldots)\ . \tag{15}$$

The three critical exponents $\eta$, $\nu$ and $\omega$ defined in this way completely specify the critical behaviour of the system. All other exponents can be expressed in terms of these [4].

The three critical exponents can be determined from the RG functions of the $\phi^4$-theory by going to the infrared-stable fixed point

$$g = g_0(\epsilon) = \sum_{k=1}^{\infty} g^{(k)}\epsilon^k \tag{16}$$

which is determined by the condition ($\beta_\epsilon \equiv \beta - \frac{\epsilon}{2} g$)

$$\beta_\epsilon'(g_0) = 0, \quad \beta_\epsilon(g_0) = [\partial\beta_\epsilon(g)/\partial g]_{g=g_0} > 0 \ . \tag{17}$$

The resulting formulas for the critical exponents are:

$$\eta = 2\gamma_2(g_0) \ , \quad 1/\nu = 2(1-\gamma_m(g_0)) \ , \quad w = 2\beta_\epsilon'(g_0) \ , \tag{18}$$

each emerging as an $\epsilon$-expansion up to order $\epsilon^5$. From (8)-(10) we therefore find:

$$\begin{aligned}
\eta(\epsilon) \;=\; & \tfrac{(n+2)\epsilon^2}{2(n+8)^2}\Big\{1 + \tfrac{\epsilon}{4(n+8)^2}[-n^2+56n+272] \\
& -\tfrac{\epsilon^2}{16(n+8)^4}[5n^4+230n^3-1124n^2-17920n-46144 \\
& \qquad + \zeta(3)(n+8)\cdot 384(5n+22)] \\
& -\tfrac{\epsilon^3}{64(n+8)^6}[13n^6+946n^5+27620n^4+121472n^3 \\
& \qquad -262528n^2-2912768n-5655552 \\
& \qquad -\zeta(3)(n+8)\cdot 16(n^5+10n^4+1220n^3-1136n^2 \\
& \qquad\qquad -68672n-171264) \\
& \qquad +\zeta(4)(n+8)^3\cdot 1152(5n+22) \\
& \qquad -\zeta(5)(n+8)^2\cdot 5120(2n^2+55n+186)]\Big\} \ ,
\end{aligned} \tag{19}$$

$$\begin{aligned}
1/\nu(\epsilon) \;=\; & 2 + \tfrac{(n+2)\epsilon}{n+8}\Big\{-1 - \tfrac{\epsilon}{2(n+8)^2}(13n+44) \\
& +\tfrac{\epsilon^2}{8(n+8)^4}\ [3n^3-452n^2-2672n-5312 \\
& \qquad +\zeta(3)(n+8)\cdot 96(5n+22)] \\
& +\tfrac{\epsilon^3}{32(n+8)^6}\ [3n^5+398n^4-12900n^3-81552n^2-219968n-357120 \\
& \qquad +\zeta(3)(n+8)\cdot 16(3n^4-194n^3+148n^2+9472n+19488) \\
& \qquad +\zeta(4)(n+8)^3\cdot 288(5n+22) \\
& \qquad -\zeta(5)(n+8)^2\cdot 1280(2n^2+55n+186)] \\
& +\tfrac{\epsilon^4}{128(n+8)^8}[3n^7-1198n^6-27484n^5-1055344n^4 \\
& \qquad -5242112n^3-5256704n^2+6999040n-626688 \\
& \qquad -\zeta(3)(n+8)\cdot 16(13n^6-310n^5+19004n^4+102400n^3 \\
& \qquad\qquad -381536n^2-2792576n-4240640) \\
& \qquad -\zeta^2(3)(n+8)^2\cdot 1024(2n^4+18n^3+981n^2+6994n+11688) \\
& \qquad +\zeta(4)(n+8)^3\cdot 48(3n^4-194n^3+148n^2+9472n+19488) \\
& \qquad +\zeta(5)(n+8)^2\cdot 256(155n^4+3026n^3+989n^2-66018n-130608)
\end{aligned}$$

$$
\begin{aligned}
&- \zeta(6)(n+8)^4 \cdot 6400(2n^2+55n+186) \\
&+ \zeta(7)(n+8)^3 \cdot 56448(14n^2+189n+526)]\Big\} \ ,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\omega(\epsilon) \ = \ & \epsilon - \tfrac{\epsilon^2}{(n+8)^2}[9n+42] \\
& + \tfrac{\epsilon^3}{4(n+8)^4}\,[33n^3+538n^2+4288n+9568 \\
& \qquad + \zeta(3)(n+8)\cdot 96(5n+22)] \\
& + \tfrac{\epsilon^4}{16(n+8)^6}[5n^5-1488n^4-46616n^3-419528n^2-1750080n-2599552 \\
& \qquad - \zeta(3)(n+8)\cdot 96(63n^3+548n^2+1916n+3872) \\
& \qquad + \zeta(4)(n+8)^3\cdot 288(5n+22) \\
& \qquad - \zeta(5)(n+8)^2\cdot 1920(2n^2+55n+186)] \\
& + \tfrac{\epsilon^5}{64(n+8)^8}[13n^7+7196n^6+240328n^5+3760776n^4 \\
& \qquad +38877056n^3+223778048n^2+660389888n+752420864 \\
& \qquad - \zeta(3)(n+8)\cdot 16(9n^6-1104n^5-11648n^4-243864n^3 \\
& \qquad\qquad -2413248n^2-9603328n-14734080) \\
& \qquad - \zeta^2(3)(n+8)^2\cdot 768(6n^4+107n^3+1826n^2+9008n+8736) \\
& \qquad - \zeta(4)(n+8)^3\cdot 288(63n^3+548n^2+1916n+3872) \\
& \qquad + \zeta(5)(n+8)^2\cdot 256(305n^4+7386n^3+45654n^2+143212n+226992) \\
& \qquad - \zeta(6)(n+8)^4\cdot 9600(2n^2+55n+186) \\
& \qquad + \zeta(7)(n+8)^3\cdot 112896(14n^2+189n+526)] \ .
\end{aligned}
\tag{21}
$$

For $n=1$, these expansions read, numerically,

$$\eta = 0.01852\,\epsilon^2 + 0.01869\,\epsilon^3 - 0.00833\,\epsilon^4 + 0.02566\,\epsilon^5 \ , \tag{22}$$

$$\tfrac{1}{\nu} = 2 - 0.333\,\epsilon - 0.1173\,\epsilon^2 + 0.1245\,\epsilon^3 - 0.307\,\epsilon^4 + 0.951\,\epsilon^5 \ , \tag{23}$$

$$\omega = \epsilon - 0.630\,\epsilon^2 + 1.618\,\epsilon^3 - 5.24\,\epsilon^4 + 20.75\,\epsilon^5 \ . \tag{24}$$

Note that $\eta^{(5)}$ has decreased by about 30 % in comparison with the (incorrect) results of Ref. [14], $\nu^{(5)}$ has increased by about 10 %, and $\omega^{(5)}$ increased by about 0.6 % in comparison with [14].

It is known that the series of the $\epsilon$-expansion are asymptotic series and special resummation techniques [20, 21] should be applied to obtain reliable estimates of the critical exponents. Although the size of the $\epsilon^5$ terms in the physical dimension

(i.e., at $\epsilon = 1$) is very large, their contribution to the exponents in the *resummed* series is very small. This is why even large relative changes of the $\epsilon^5$ coefficients turn out not to change the critical exponents [22] within the accuracy of previous determinations [14, 23].

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